Gradient estimates and entropy formulae of porous medium and fast diffusion equations for the Witten Laplacian

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Abstract. We consider gradient estimates to positive solutions of porous medium equations and fast diffusion equations:

$$u_t = \Delta_\phi(u^p)$$

associated with the Witten Laplacian on Riemannian manifolds. Under the assumption that the m-dimensional Bakry-Emery Ricci curvature is bounded from below, we obtain gradient estimates which generalize the results in [20] and [13]. Moreover, inspired by X. -D. Li's work in [19] we also study the entropy formulae introduced in [20] for porous medium equations and fast diffusion equations associated with the Witten Laplacian. We prove monotonicity theorems for such entropy formulae on compact Riemannian manifolds with non-negative m-dimensional Bakry-Emery Ricci curvature.

Keywords. porous medium equation, fast diffusion equation, entropy formulae, Witten Laplacian
Mathematics Subject Classification. Primary 35B45, Secondary 35K55

1 Introduction

Let (M^n, g) be an n-dimensional complete Riemannian manifold. Li and Yau [16] studied positive solutions of the heat equation

$$u_t = \Delta u \tag{1.1}$$

and obtained the following gradient estimates:

Theorem A(Li-Yau [16]). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution of (1.1) on $B_p(2R) \times [0, T]$. Then on $B_p(R)$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{C(n)\alpha^2}{R^2}\left(\frac{\alpha^2}{\alpha-1} + \sqrt{K}R\right) + \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t}, \tag{1.2}$$

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where $\alpha > 1$ is a constant. Moreover, when $R \to \infty$, (1.2) *yields the following estimate on complete noncompact Riemannian manifold (M^n, g):*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}. \tag{1.3}$$

Recently, J. F. Li and X. J. Xu [15] obtained new Li-Yau type gradient estimates for positive solutions of the heat equation (1.1) on Riemannian manifolds. For the related research and some improvements on Li-Yau type gradient estimates of the equation (1.1), see [2, 9, 12, 18, 27, 28] and the references therein. The equation

$$u_t = \Delta(u^p) \tag{1.4}$$

with $p > 1$ is called the porous medium equation, which is a nonlinear version of the classical heat equation. For various values of $p > 1$, it has arisen in different applications to model diffusive phenomena (see [1,20,30] and the references therein). The equation (1.4) with $p \in (0, 1)$ is called the fast diffusion equation, which appears in plasma physics and in geometric flows. However, there are marked differences between the porous medium equations and the fast diffusion equation, see [8, 29]. For gradient estimates of (1.4), see [1, 13, 30, 34].

In [20], Lu, Ni, Vázquez and Villani studied gradient estimates of (1.4) and proved the following results (see Theorem 3.3 in [20]):

Theorem B(P. Lu, L. Ni, J. Vázquez, C.Villani [20]). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to* (1.4) *with $p > 1$. Let $v = \frac{p}{p-1} u^{p-1}$ and $M = (p-1)\max_{B_p(2R)\times[0,T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have*

$$\begin{aligned}\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq & \frac{C(n)Ma\alpha^2}{R^2}\left(\frac{\alpha^2}{\alpha - 1}\frac{ap^2}{p-1} + (1 + \sqrt{K}R)\right) \\ & + \frac{\alpha^2}{\alpha - 1} aMK + \frac{a\alpha^2}{t},\end{aligned} \tag{1.5}$$

where $a = \frac{n(p-1)}{n(p-1)+2}$. Moreover, when $R \to \infty$, (1.5) *yields the following estimate on complete noncompact Riemannian manifold (M^n, g):*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} aMK + \frac{a\alpha^2}{t}. \tag{1.6}$$

Now, we rewrite the inequality (1.6) as

$$|\nabla v|^2 - \alpha v_t \leq \frac{\alpha^2}{\alpha - 1} aMKv + \frac{a\alpha^2 v}{t}. \tag{1.7}$$

Since $(p-1)v = pu^{p-1}$, we have $(p-1)v \to 1$ as $p \to 1$. Hence, $M \to 1$,

$$\begin{aligned}|\nabla v|^2 &\to \frac{|\nabla u|^2}{u^2}, \\ v_t &\to \frac{u_t}{u}, \\ av &\to \frac{n}{2},\end{aligned}$$

as $p \to 1$. As a result, (1.7) becomes the inequality (1.3) in Theorem A of Li-Yau. Therefore, for complete noncompact Riemannian manifold (M^n, g), the estimate (1.6) in Theorem B of Lu, Ni, Vázquez and Villani reduces to the estimate (1.3) in Theorem A of Li-Yau when $p \to 1$.

Let $\phi \in C^2(M^n)$. The Witten Laplacian associated with ϕ is defined by

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla$$

which is symmetric with respect to the $L^2(M^n)$ inner product under the weighted measure

$$d\mu = e^{-\phi} dv,$$

that is,

$$\int_{M^n} u\Delta_\phi v \, d\mu = -\int_{M^n} \nabla u \nabla v \, d\mu = \int_{M^n} v\Delta_\phi u \, d\mu, \quad \forall\, u, v \in C_0^\infty(M^n).$$

The m-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian is given by

$$\mathrm{Ric}_\phi^m = \mathrm{Ric} + \nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi,$$

where $m > n$ and $m = n$ if and only if ϕ is a constant. Define

$$\mathrm{Ric}_\phi = \mathrm{Ric} + \nabla^2\phi.$$

Then Ric_ϕ can be seen as the ∞-dimensional Bakry-Emery Ricci curvature. In this paper, we study the following equation associated with the Witten Laplacian:

$$u_t = \Delta_\phi(u^p) \tag{1.8}$$

with $p > 0$ and $p \neq 1$. For $p > 1$ and $p \in (0, 1)$, we derive estimates of Lu, Ni, Vázquez and Villani and Davies's type estimate. Moreover, for $p > 1$, we obtain Hamilton's type estimate and estimates of J. F. Li and X. J. Xu. In particular, our results generalize the ones in [13].

First we consider gradient estimates of (1.8) under the assumption that the m-dimensional Bakry-Emery Ricci curvature is bounded from blew, and obtain the following results:

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_\phi^m(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to the porous medium equation* (1.8) *with $p > 1$. Let $v = \frac{p}{p-1} u^{p-1}$ and $M = (p-1)\max_{B_p(2R)\times[0,T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq & \tilde{a}\alpha^2 M \frac{C(m)}{R^2} \left\{ \frac{\alpha^2}{\alpha - 1} \frac{\tilde{a}p^2}{p-1} + \left(1 + \sqrt{K} R \coth(\sqrt{K} R)\right) \right\} \\ & + \frac{\alpha^2}{(\alpha-1)} \tilde{a} M K + \frac{\tilde{a}\alpha^2}{t}, \end{aligned} \tag{1.9}$$

where $\tilde{a} = \frac{m(p-1)}{m(p-1)+2}$. Moreover, when $R \to \infty$, (1.9) *yields the following estimate on complete noncompact Riemannian manifold (M^n, g):*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} \tilde{a} M K + \frac{\tilde{a}\alpha^2}{t}. \tag{1.10}$$

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_\phi^m(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to the fast diffusion equation (1.8) with $p \in (1-\frac{2}{m}, 1)$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (1-p)\max_{B_p(2R)\times[0,T]}(-v)$. Then for any $0<\alpha<1$, on the ball $B_p(R)$, we have*

$$\begin{aligned} -\frac{|\nabla v|^2}{v} + \alpha\frac{v_t}{v} \leq & \frac{(-\tilde{a})\alpha^2 M}{A(\varepsilon_1,\varepsilon_2)}\frac{C(m)}{R^2}\left\{\frac{(-\tilde{a})\alpha^2 p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + \left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right\} \\ & + \frac{(-\tilde{a})\alpha^2 MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})A(\varepsilon_1,\varepsilon_2)}} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1,\varepsilon_2)t}, \end{aligned} \tag{1.11}$$

where $\tilde{a} = \frac{m(p-1)}{m(p-1)+2}$ and positive constants $\varepsilon_1, \varepsilon_2 \in (0,1)$ satisfying

$$A(\varepsilon_1,\varepsilon_2) := [1-\tilde{a}(1-\alpha)] - \frac{(1+\varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1-\varepsilon_1)(1-\alpha-\tilde{a})} > 0.$$

When $R \to \infty$ and $\alpha \to 1$, (1.11) yields the following estimate on complete noncompact Riemannian manifold (M^n, g) with $\mathrm{Ric}_\phi^m \geq 0$:

$$-\frac{|\nabla v|^2}{v} + \frac{v_t}{v} \leq -\frac{\tilde{a}}{t}. \tag{1.12}$$

Remark 1.1. Clearly, our estimate (1.10) reduces to (1.6) of Lu, Ni, Vázquez and Villani (see [20]) by letting $m = n$. Moreover, for $p \in (0,1)$, Theorem 4.1 in [20] of Lu, Ni, Vázquez and Villani can be obtained from our Theorem 1.2 by taking $m = n$.

Theorem 1.3. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_\phi^m(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to the porous medium equation (1.8) with $p > 1$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (p-1)\max_{B_p(2R)\times[0,T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq & \tilde{a}\alpha^2\left\{\frac{\tilde{a}^{\frac{1}{2}}\alpha p M^{\frac{1}{2}}}{(p-1)^{\frac{1}{2}}(\alpha-1)^{\frac{1}{2}}}\frac{C(m)}{R} + \left[\frac{1}{t} + \frac{MK}{2(\alpha-1)}\right.\right. \\ & \left.\left. + M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right]^{\frac{1}{2}}\right\}^2, \end{aligned} \tag{1.13}$$

where $\tilde{a} = \frac{m(p-1)}{m(p-1)+2}$. Moreover, when $R \to \infty$, (1.13) yields the following estimate on complete noncompact Riemannian manifold:

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq \frac{\alpha^2}{2(\alpha-1)}\tilde{a}MK + \frac{\tilde{a}\alpha^2}{t}. \tag{1.14}$$

Theorem 1.4. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_\phi^m(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to the fast diffusion equation (1.8) with $p \in (1-\frac{2}{m}, 1)$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (1-p)\max_{B_p(2R)\times[0,T]}(-v)$. Then for any*

$0 < \alpha < 1$, *on the ball* $B_p(R)$, *we have*

$$-\frac{|\nabla v|^2}{v}+\alpha\frac{v_t}{v}\leq\Bigg\{C(\tilde{a},\alpha)\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}\frac{C}{R}+\Big[\Big(\frac{\alpha^2}{2(1-\alpha)}+2(1-\tilde{a})\Big)MK+\frac{1-\alpha-\tilde{a}}{t}$$
$$+(1-p)(1-\alpha-\tilde{a})M\frac{C(m)}{R^2}\Big(1+\sqrt{K}R\coth(\sqrt{K}R)\Big)\Big]^{\frac{1}{2}}\Bigg\}^2, \tag{1.15}$$

where $\tilde{a}=\frac{m(p-1)}{m(p-1)+2}$. *When* $R\to\infty$, (1.15) *yields the following estimate on complete noncompact Riemannian manifold* (M^n,g)*:*

$$-\frac{|\nabla v|^2}{v}+\alpha\frac{v_t}{v}\leq\Big(\frac{\alpha^2}{2(1-\alpha)}+2(1-\tilde{a})\Big)MK+\frac{1-\alpha-\tilde{a}}{t}. \tag{1.16}$$

Remark 1.2. Our Theorem 1.3 reduces to Theorem 1.1 of [13] by letting $m=n$ and the estimate (1.14) improves (1.10) on complete noncompact Riemannian manifolds. For complete noncompact Riemannian manifolds with $p\in(0,1)$, Lu, Ni, Vázquez and Villani [20] proved (see Corollary 4.2 in [20]) the following results: If $\mathrm{Ric}\geq 0$, then

$$-\frac{|\nabla v|^2}{v}+\frac{v_t}{v}\leq-\frac{a}{t}; \tag{1.17}$$

If $\mathrm{Ric}\geq -K$ and $0<\alpha<1$, then for any $\varepsilon>0$ satisfying $C(a,\alpha,\varepsilon):=1+(-a)(1-\alpha)-\frac{(1-\alpha)(1-a)^2}{(1-\alpha)-a-(1-\alpha)\varepsilon^2}>0$,

$$-\frac{|\nabla v|^2}{v}+\alpha\frac{v_t}{v}\leq\frac{(-a)\alpha^2}{C(a,\alpha,\varepsilon)}\Big(\frac{1}{t}+\frac{\sqrt{C(a,\alpha,\varepsilon)}}{(1-\alpha)\varepsilon}MK\Big). \tag{1.18}$$

Obviously, our estimate (1.16) reduces to (1.17) of Lu, Ni, Vázquez and Villani when $m=n$ and $\alpha\to 1$. Moreover, (1.16) is independent of ε.

Theorem 1.5. *Let* (M^n,g) *be a complete Riemannian manifold with* $\mathrm{Ric}^m_\phi(B_p(2R))\geq -K$, $K\geq 0$. *Suppose that* u *is a positive solution to the porous medium equation* (1.8) *with* $p>1$. *Let* $v=\frac{p}{p-1}u^{p-1}$ *and* $M=(p-1)\max_{B_p(2R)\times[0,T]}v$. *Then for any* $\alpha>1$, *on the ball* $B_p(R)$, *we have*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}\leq\tilde{a}\alpha^2(t)M\frac{C(m)}{R^2}\left(\frac{p^2\tilde{a}\alpha^2(t)}{2(p-1)(\alpha(t)-1)}+3+\sqrt{K}R\coth(\sqrt{K}R)\right)+\frac{\tilde{a}\alpha^2(t)}{t}, \tag{1.19}$$

where $\tilde{a}=\frac{m(p-1)}{m(p-1)+2}$ *and* $\alpha(t)=e^{2MKt}$. *Moreover, when* $R\to\infty$, (1.19) *yields the following estimate on complete noncompact Riemannian manifold:*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}\leq\frac{\tilde{a}\alpha^2(t)}{t}. \tag{1.20}$$

Remark 1.3. Our Theorem 1.5 becomes Theorem 1.2 in [13] as long as we let $m=n$.

Theorem 1.6. *Let* (M^n,g) *be a complete Riemannian manifold with* $\mathrm{Ric}^m_\phi(B_p(2R))\geq -K$, $K\geq 0$. *Suppose that* u *is a positive solution to the porous medium equation* (1.8)

with $p>1$. *Let* $v=\frac{p}{p-1}u^{p-1}$ *and* $M=(p-1)\max_{B_p(2R)\times[0,T]}v$. *Then on the ball* $B_p(R)$, *we have*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}-\varphi(t)\leq\tilde{a}M\frac{C(m)}{R^2}\Big\{1+\sqrt{K}R\coth(\sqrt{K}R)+\frac{\tilde{a}p^2}{(p-1)\tanh(MKt)}\Big\},\tag{1.21}$$

where $\tilde{a}=\frac{m(p-1)}{m(p-1)+2}$, $\alpha(t)$ *and* $\varphi(t)$ *are given by*

$$\begin{aligned}\varphi(t)=&\tilde{a}MK\{\coth(MKt)+1\},\\ \alpha(t)=&1+\frac{\cosh(MKt)\sinh(MKt)-MKt}{\sinh^2(MKt)}.\end{aligned}\tag{1.22}$$

Moreover, when $R\to\infty$, (1.21) *yields the following estimate on complete noncompact Riemannian manifold:*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}-\varphi(t)\leq 0.\tag{1.23}$$

Theorem 1.7. *Let* (M^n,g) *be a complete Riemannian manifold with* $\mathrm{Ric}_\phi^m(B_p(2R))\geq -K$, $K\geq 0$. *Suppose that* u *is a positive solution to the porous medium equation* (1.8) *with* $p>1$. *Let* $v=\frac{p}{p-1}u^{p-1}$ *and* $M=(p-1)\max_{B_p(2R)\times[0,T]}v$. *Then on the ball* $B_p(R)$, *we have*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}-\varphi(t)\leq\tilde{a}\alpha^2(t)M\frac{C(m)}{R^2}\Big\{1+\sqrt{K}R\coth(\sqrt{K}R)+\frac{\tilde{a}p^2\alpha^2(t)}{(p-1)\tanh(MKt)}\Big\},\tag{1.24}$$

where $\tilde{a}=\frac{m(p-1)}{m(p-1)+2}$, $\alpha(t)$ *and* $\varphi(t)$ *are given by*

$$\begin{aligned}\varphi(t)=&\frac{\tilde{a}}{t}+\tilde{a}MK+\frac{\tilde{a}}{3}(MK)^2t,\\ \alpha(t)=&1+\frac{2}{3}MKt.\end{aligned}\tag{1.25}$$

Moreover, when $R\to\infty$, (1.21) *yields the following estimate on complete noncompact Riemannian manifold:*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}-\varphi(t)\leq 0.\tag{1.26}$$

Remark 1.4. Our Theorems 1.6 and 1.7 reduce to Theorems 1.3 and 1.4 in [13] by taking $m=n$, respectively. Moreover, when t is small enough, $\alpha(t),\varphi(t)$ defined by (1.22) and (1.25) both satisfy $\alpha(t)\to 1$ and $\varphi(t)\leq 2\tilde{a}MK+\frac{\tilde{a}}{t}$. Hence, (1.23) and (1.26) show

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}\leq 2\tilde{a}MK+\frac{\tilde{a}}{t}.\tag{1.27}$$

Clearly, for t small enough, (1.27) is better than (1.10). Therefore, (1.23) and (1.26) improve (1.10) on complete noncompact Riemannian manifolds in this sense.

Denote by R the scalar curvature of the metric g. In [24], Perelman introduced the $\mathcal{W}$-entropy functional as follows:

$$\mathcal{W}(g,f,\tau)=\int_{M^n}[\tau(R+|\nabla f|^2)+f-n]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\,dv,\tag{1.28}$$

where τ is a positive scale parameter and $f \in C^\infty(M^n)$ satisfies

$$\int_{M^n} \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\, dv = 1.$$

By [24], we know that the $\mathcal{W}$-entropy is monotone increasing under the Ricci flow, and its critical points are given by gradient shrinking solitons. In [21, 22], Ni considered the $\mathcal{W}$-entropy for the linear heat equation

$$u_\tau = \Delta u \tag{1.29}$$

on complete Riemannian manifolds. More precisely, for the $\mathcal{W}$-entropy associated with (1.29):

$$\mathcal{W}(g, f, \tau) = \int_{M^n} [\tau|\nabla f|^2 + f - n]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\, dv, \tag{1.30}$$

where $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ is a positive solution to (1.29) and $\int_{M^n} u\, dv = 1$, Ni [21] proved

$$\frac{d}{d\tau}\mathcal{W}(g, f, \tau) = -2\int_{M^n} \tau\Big(\Big|\nabla^2 f - \frac{g}{2\tau}\Big|^2 + \mathrm{Ric}(\nabla f, \nabla f)\Big)u\, dv. \tag{1.31}$$

In particular, if the Ricci curvature is non-negative, then $\mathcal{W}$-entropy defined by (1.31) is monotone non-increasing on complete Riemannian manifolds. For the research of the monotonicity of $\mathcal{W}$-entropy to other geometric heat flows on Riemannian manifolds, see [10, 14, 20–22]. In [19], Li studied the $\mathcal{W}_m$-entropy associated with the Witten Laplacian to the linear heat equation

$$u_\tau = \Delta_\phi u \tag{1.32}$$

on complete Riemannian manifolds satisfying the μ-bounded geometry condition. More precisely, for the $\mathcal{W}_m$-entropy associated with (1.32):

$$\mathcal{W}_m(g, f, \tau) = \int_{M^n} [\tau|\nabla f|^2 + f - m]\frac{e^{-f}}{(4\pi\tau)^{\frac{m}{2}}}\, d\mu, \tag{1.33}$$

where $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{m}{2}}}$ is a positive solution to (1.32), Li [19] proved that if there exist two constants $m > n$ and $K \geq 0$ such that $\mathrm{Ric}_\phi^m \geq -K$, then

$$\begin{aligned}\frac{d}{d\tau}\mathcal{W}_m(g, f, \tau) =& -2\int_{M^n} \tau\Big(\Big|\nabla^2 f - \frac{g}{2\tau}\Big|^2 + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\Big)u\, d\mu\\ &- \frac{2}{m-n}\int_{M^n} \tau\Big(\nabla\phi\nabla f + \frac{m-n}{2\tau}\Big)^2 u\, d\mu.\end{aligned} \tag{1.34}$$

In particular, if the $\mathrm{Ric}_\phi^m \geq 0$, then $\mathcal{W}_m(g, f, \tau)$ is non-increasing along the heat equation (1.32). For the study to the Witten Laplacian associated with the m-dimensional Bakry-Emery Ricci curvature on complete Riemannian manifolds, see [3–5,11,18,23,25,26,31–33]. Let u be a positive solution to (1.4), and let $v = \frac{p}{p-1}u^{p-1}$. In [20], Lu, Ni, Vázquez and Villani introduced the following:

$$\mathcal{N}_p(g, u, t) = -t^a \int_{M^n} uv\, dv$$

and

$$\mathcal{W}_p(g,u,t)=\frac{d}{dt}[t\mathcal{N}_p(g,u,t)]=t^{a+1}\int\limits_{M^n}\Big(p\frac{|\nabla v|^2}{v}-\frac{a+1}{t}\Big)uv\,dv, \tag{1.35}$$

where $a=\frac{n(p-1)}{n(p-1)+2}$. They proved that if M^n is compact, then

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_p(g,u,t)=&-2(p-1)t^{a+1}\int\limits_{M^n}\Big(\Big|\nabla^2 v+\frac{g}{[n(p-1)+2]t}\Big|^2+\mathrm{Ric}(\nabla v,\nabla v)\Big)uv\,dv\\&-2t^{a+1}\int\limits_{M^n}\Big((p-1)\Delta v+\frac{a}{t}\Big)^2 uv\,dv.\end{aligned} \tag{1.36}$$

In particular, if the Ricci curvature is non-negative, then the entropy defined in (1.35) is monotone non-increasing on compact Riemannian manifolds when $p>1$. For $p<1$, using the Cauchy-Schwarz inequality, they proved from (1.36) that

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_p(g,u,t)\le&-2t^{a+1}\int\limits_{M^n}\Big[\frac{n(p-1)+1}{n(p-1)}\Big((p-1)\Delta v+\frac{a}{t}\Big)^2\\&+(p-1)\mathrm{Ric}(\nabla v,\nabla v)\Big]uv\,dv.\end{aligned} \tag{1.37}$$

Clearly, if the Ricci curvature is non-negative and $p\in(1-\frac{1}{n},1)$, then (1.37) shows that $\frac{d}{dt}\mathcal{W}_p(g,u,t)\le 0$ and the entropy defined in (1.35) is monotone non-increasing on compact Riemannian manifolds.

Inspired by [19], in this paper we also study the $\mathcal{W}_{p,m}$-entropy associated with the Witten Laplacian to the equation (1.8) on compact Riemannian manifolds with $p>0$ and $p\neq 1$. First we define

$$\mathcal{N}_{p,m}(g,u,t)=-t^{\tilde a}\int\limits_{M^n}uv\,d\mu \tag{1.38}$$

and the $\mathcal{W}_{p,m}$-entropy is defined by

$$\mathcal{W}_{p,m}(g,u,t)=\frac{d}{dt}[t\mathcal{N}_{p,m}(g,u,t)], \tag{1.39}$$

where $\tilde a=\frac{m(p-1)}{m(p-1)+2}$. Under the m-dimensional Bakry-Emery Ricci curvature is bounded from below, we prove the following:

Theorem 1.8. *Let (M^n,g) be a compact Riemannian manifold. If u is a positive solution to the porous medium equation (1.8) with $p>1$, then*

$$\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)=-t^{\tilde a}\int\limits_{M^n}\Big((p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big)uv\,d\mu, \tag{1.40}$$

where $v=\frac{p}{p-1}u^{p-1}$ and $\tilde a=\frac{m(p-1)}{m(p-1)+2}$. In particular, if $\mathrm{Ric}_\phi^m\ge 0$, then $\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)\le 0$ and $\mathcal{N}_{p,m}(g,u,t)$ is monotone non-increasing in t. Moreover,

$$\mathcal{W}_{p,m}(g,u,t)=t^{\tilde a+1}\int\limits_{M^n}\Big(p\frac{|\nabla v|^2}{v}-\frac{\tilde a+1}{t}\Big)uv\,d\mu \tag{1.41}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t) =& -2(p-1)t^{\tilde{a}+1}\int\limits_{M^n}\Big\{\Big|\nabla^2 v+\frac{g}{[m(p-1)+2]t}\Big|^2\\&+\frac{1}{m-n}\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2+\mathrm{Ric}_\phi^m(\nabla v,\nabla v)\Big\}uv\,d\mu\\&-2t^{\tilde{a}+1}\int\limits_{M^n}\Big|(p-1)\Delta_\phi v+\frac{\tilde{a}}{t}\Big|^2 uv\,d\mu.\end{aligned}\tag{1.42}$$

In particular, if $\mathrm{Ric}_\phi^m\geq 0$*, then* $\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t)\leq 0$ *and* $\mathcal{W}_{p,m}(g,u,t)$ *is monotone non-increasing in* t*.*

Theorem 1.9. *If* u *is a positive solution to the fast diffusion equation* (1.8) *with* $p\in(0,1)$*, then*

$$\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)=-t^{\tilde{a}}\int\limits_{M^n}\Big((p-1)\Delta_\phi v+\frac{\tilde{a}}{t}\Big)uv\,d\mu,\tag{1.43}$$

where $v=\frac{p}{p-1}u^{p-1}$ *and* $\tilde{a}=\frac{m(p-1)}{m(p-1)+2}$*. In particular, if* $\mathrm{Ric}_\phi^m\geq 0$ *and* $p\in(1-\frac{2}{m},1)$*, then* $\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)\leq 0$ *and* $\mathcal{N}_{p,m}(g,u,t)$ *is monotone non-increasing in* t*. Moreover,*

$$\mathcal{W}_{p,m}(g,u,t)=t^{\tilde{a}+1}\int\limits_{M^n}\Big(p\frac{|\nabla v|^2}{v}-\frac{\tilde{a}+1}{t}\Big)uv\,d\mu\tag{1.44}$$

and for any positive constant $\varepsilon\geq m-n$ *and* $1-\frac{1}{n+\varepsilon}\leq p\leq 1-\frac{m-n}{m\varepsilon}$*,*

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t)\leq& 2t^{\tilde{a}+1}\int\limits_{M^n}\Big\{(1-p)\mathrm{Ric}_\phi^m(\nabla v,\nabla v)\\&+\Big(\frac{1-n(1-p)}{n(1-p)}-\frac{\varepsilon}{n}\Big)\Big|(p-1)\Delta_\phi v+\frac{\tilde{a}}{t}\Big|^2\\&+\Big(\frac{m(1-p)}{n(m-n)}-\frac{1}{n\varepsilon}\Big)\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2\Big\}uv\,d\mu.\end{aligned}\tag{1.45}$$

In particular, if $\mathrm{Ric}_\phi^m\geq 0$*, then* $\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t)\leq 0$ *and* $\mathcal{W}_{p,m}(g,u,t)$ *is monotone non-increasing in* t*.*

Remark 1.5. In particular, if $m=n$, then we have that ϕ is a constant. Then (1.42) becomes (5.6) of Lu, Ni, Vázquez and Villani in [20]. By letting $m=n$ and $\varepsilon\to 0$, (1.45) becomes (1.37), which is Corollary 5.10 in [20].

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2 Proofs of Theorem 1.1 and 1.2

Let $v=\frac{p}{p-1}u^{p-1}$. By virtue of the equation (1.8), we have $v_t=(p-1)v\Delta_\phi v+|\nabla v|^2$ which is equivalent to

$$\frac{v_t}{v}=(p-1)\Delta_\phi v+\frac{|\nabla v|^2}{v}.\tag{2.1}$$

Lemma 2.1. *As in [20], we introduce the following differential operator*

$$\mathcal{L} = \partial_t - (p-1)v\Delta_\phi.$$

Let $F = \frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} - \varphi$, *where* $\alpha = \alpha(t)$ *and* $\varphi = \varphi(t)$ *are functions depending on* t.

(1) If $p > 1$, *then*

$$\begin{aligned}\mathcal{L}(F) \leq & -\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 - 2(p-1)\mathrm{Ric}_\phi^m(\nabla v, \nabla v) + 2p\nabla v\nabla F \\ & + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha'\frac{v_t}{v} - \varphi';\end{aligned} \tag{2.2}$$

(2) If $p \in (0,1)$, *then*

$$\begin{aligned}\mathcal{L}(F) \geq & -\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 - 2(p-1)\mathrm{Ric}_\phi^m(\nabla v, \nabla v) + 2p\nabla v\nabla F \\ & + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha'\frac{v_t}{v} - \varphi',\end{aligned} \tag{2.3}$$

where $\tilde{a} = \frac{m(p-1)}{m(p-1)+2}$.

Proof. We only give the proof to the case that $p > 1$. The proof to $p < 1$ is similar, so we omit it here.

By a direct calculation, we have

$$\mathcal{L}\Big(\frac{f}{g}\Big) = \frac{1}{g}\mathcal{L}(f) - \frac{f}{g^2}\mathcal{L}(g) + 2(p-1)v\nabla\Big(\frac{f}{g}\Big)\nabla\log g, \qquad \forall\ f, g \in C^\infty(M). \tag{2.4}$$

Using (2.1) we obtain

$$\mathcal{L}(v_t) = (p-1)v_t\Delta_\phi v + 2\nabla v\nabla v_t. \tag{2.5}$$

It is well known that for the m-dimensional Bakry-Emery Ricci curvature, we have the following Bochner formula (for the elementary proof, see [17, 18]):

$$\begin{aligned}\frac{1}{2}\Delta_\phi(|\nabla w|^2) =& |\nabla^2 w|^2 + \nabla w\nabla\Delta_\phi w + \mathrm{Ric}_\phi(\nabla w, \nabla w) \\ \geq & \frac{1}{n}|\Delta w|^2 + \nabla w\nabla\Delta_\phi w + \mathrm{Ric}_\phi(\nabla w, \nabla w) \\ \geq & \frac{1}{m}|\Delta_\phi w|^2 + \nabla w\nabla\Delta_\phi w + \mathrm{Ric}_\phi^m(\nabla w, \nabla w).\end{aligned}$$

It follows from $p > 1$ that

$$\begin{aligned}\mathcal{L}(|\nabla v|^2) \leq & 2\nabla v\nabla v_t - 2(p-1)v\Big(\frac{1}{m}|\Delta_\phi v|^2 + \nabla v\nabla\Delta_\phi v + \mathrm{Ric}_\phi^m(\nabla v, \nabla v)\Big) \\ =& 2\nabla v\nabla[(p-1)v\Delta_\phi v + |\nabla v|^2] - 2(p-1)v\Big(\frac{1}{m}|\Delta_\phi v|^2 \\ & + \nabla v\nabla\Delta_\phi v + \mathrm{Ric}_\phi^m(\nabla v, \nabla v)\Big) \\ =& 2(p-1)|\nabla v|^2\Delta_\phi v + 2\nabla v\nabla(|\nabla v|^2) - \frac{2(p-1)}{m}v(\Delta_\phi v)^2 \\ & - 2(p-1)v\mathrm{Ric}_\phi^m(\nabla v, \nabla v).\end{aligned} \tag{2.6}$$

Applying (2.5) and (2.6) into (2.4) yields

$$\mathcal{L}\left(\frac{v_t}{v}\right) = (p-1)\frac{v_t}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla v_t - \frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2(p-1)v\nabla\left(\frac{v_t}{v}\right)\nabla(\log v),$$

$$\begin{aligned}\mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) \leq& 2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \frac{2(p-1)}{m}(\Delta_\phi v)^2\\ &- 2(p-1)\mathrm{Ric}_\phi^m(\nabla v,\nabla v) - \frac{|\nabla v|^4}{v^2} + 2(p-1)v\nabla\left(\frac{|\nabla v|^2}{v}\right)\nabla(\log v)\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{L}(F) =& \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) - \alpha\mathcal{L}\left(\frac{v_t}{v}\right) - \alpha'\frac{v_t}{v} - \varphi'\\ \leq& 2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \frac{2(p-1)}{m}(\Delta_\phi v)^2\\ &- 2(p-1)\mathrm{Ric}_\phi^m(\nabla v,\nabla v) - \frac{|\nabla v|^4}{v^2} + 2(p-1)v\nabla\left(\frac{|\nabla v|^2}{v}\right)\nabla(\log v)\\ &- \alpha(p-1)\frac{v_t}{v}\Delta_\phi v - \alpha\frac{2}{v}\nabla v\nabla v_t + \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} - 2\alpha(p-1)v\nabla\left(\frac{v_t}{v}\right)\nabla(\log v)\\ &- \alpha'\frac{v_t}{v} - \varphi'.\end{aligned} \tag{2.7}$$

Noticing

$$2(p-1)v\nabla(\frac{|\nabla v|^2}{v})\nabla(\log v) - 2\alpha(p-1)v\nabla(\frac{v_t}{v})\nabla(\log v) = 2(p-1)\nabla v\nabla F,$$

$$\frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \alpha\frac{2}{v}\nabla v\nabla v_t = \frac{2}{v}\nabla v\nabla[(F+\varphi)v] = 2(F+\varphi)\frac{|\nabla v|^2}{v} + 2\nabla v\nabla F,$$

we have

$$\begin{aligned}&2(p-1)v\nabla\left(\frac{|\nabla v|^2}{v}\right)\nabla(\log v) - 2\alpha(p-1)v\nabla\left(\frac{v_t}{v}\right)\nabla(\log v) + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \alpha\frac{2}{v}\nabla v\nabla v_t\\ &= 2p\nabla v\nabla F + 2(F+\varphi)\frac{|\nabla v|^2}{v}\\ &= 2p\nabla v\nabla F + 2\left(\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}\right)\frac{|\nabla v|^2}{v}.\end{aligned} \tag{2.8}$$

On the other hand, using (2.1) again, we have

$$\begin{aligned}&2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v - \frac{|\nabla v|^4}{v^2} - \alpha(p-1)\frac{v_t}{v}\Delta_\phi v + \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\ &= 2\frac{|\nabla v|^2}{v}\left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}\right) - \frac{|\nabla v|^4}{v^2} - \alpha\frac{v_t}{v}\left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}\right) + \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\ &= (2\alpha+2)\frac{v_t}{v}\frac{|\nabla v|^2}{v} - 3\frac{|\nabla v|^4}{v^2} - \alpha\left(\frac{v_t}{v}\right)^2.\end{aligned} \tag{2.9}$$

Combining (2.8) with (2.9) gives

$$\begin{aligned}
&2(p-1)v\nabla\left(\frac{|\nabla v|^2}{v}\right)\nabla(\log v)-2\alpha(p-1)v\nabla\left(\frac{v_t}{v}\right)\nabla(\log v)+\frac{2}{v}\nabla v\nabla(|\nabla v|^2)\\
&\quad-\alpha\frac{2}{v}\nabla v\nabla v_t+2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v-\frac{|\nabla v|^4}{v^2}-\alpha(p-1)\frac{v_t}{v}\Delta_\phi v+\alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\
&=2p\nabla v\nabla F-\left(\frac{v_t}{v}-\frac{|\nabla v|^2}{v}\right)^2+(1-\alpha)\left(\frac{v_t}{v}\right)^2\\
&=2p\nabla v\nabla F-[(p-1)\Delta_\phi v]^2+(1-\alpha)\left(\frac{v_t}{v}\right)^2.
\end{aligned}\tag{2.10}$$

Putting (2.10) into (2.7) yields

$$\begin{aligned}
\mathcal{L}(F)\le&-\frac{2(p-1)}{m}(\Delta_\phi v)^2-2(p-1)\mathrm{Ric}_\phi^m(\nabla v,\nabla v)+2p\nabla v\nabla F\\
&-[(p-1)\Delta_\phi v]^2+(1-\alpha)(\frac{v_t}{v})^2-\alpha'\frac{v_t}{v}-\varphi'\\
=&-\frac{1}{\tilde a}[(p-1)\Delta_\phi v]^2-2(p-1)\mathrm{Ric}_\phi^m(\nabla v,\nabla v)+2p\nabla v\nabla F\\
&+(1-\alpha)(\frac{v_t}{v})^2-\alpha'\frac{v_t}{v}-\varphi',
\end{aligned}$$

which completes the proof of (1) in Lemma 2.1. □

Proof of Theorem 1.1. Let ξ be a cut-off function such that $\xi(r)=1$ for $r\le 1$, $\xi(r)=0$ for $r\ge 2$, $0\le\xi(r)\le 1$, and

$$0\ge\xi'(r)\ge -c_1\xi^{\frac{1}{2}}(r),$$
$$\xi''(r)\ge -c_2,$$

for positive constants c_1 and c_2. Denote by $\rho(x)=d(x,p)$ the distance between x and p in M^n. Let

$$\psi(x)=\xi\left(\frac{\rho(x)}{R}\right).$$

Making use of an argument of Calabi [6] (see also Cheng and Yau [7]), we can assume without loss of generality that the function ψ is smooth in $B_p(2R)$. Then, we have

$$\frac{|\nabla\psi|^2}{\psi}\le\frac{C}{R^2}.\tag{2.11}$$

By the comparison theorem with respect to the Witten Laplacian (see p. 1324, [18])

$$\Delta_\phi\rho\ge\sqrt{(m-1)K}\coth\left(\sqrt{\frac{K}{m-1}}\;\rho\right),$$

we have

$$\Delta_\phi\psi=\frac{\xi'\Delta_\phi\rho}{R}+\frac{\xi''|\nabla\rho|^2}{R^2}\ge-\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right).\tag{2.12}$$

Define $\tilde F=\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}$, where $\alpha>1$ is a constant. Under the assumption that $\mathrm{Ric}_\phi^m\ge -K$, (2.2) shows that

$$\begin{aligned}
\mathcal{L}(\tilde F)\le&-\frac{1}{\tilde a}[(p-1)\Delta_\phi v]^2+2(p-1)K|\nabla v|^2+2p\nabla v\nabla\tilde F\\
\le&-\frac{1}{\tilde a}[(p-1)\Delta_\phi v]^2+2MK\frac{|\nabla v|^2}{v}+2p\nabla v\nabla\tilde F.
\end{aligned}\tag{2.13}$$

Define $G = t\psi\tilde{F}$. Next we will apply maximum principle to G on $B_p(2R)\times[0,T]$. Assume G achieves its maximum at the point $(x_0,s)\in B_p(2R)\times[0,T]$ and assume $G(x_0,s)>0$ (otherwise the proof is trivial), which implies $s>0$. Then at the point (x_0,s), it holds that

$$\mathcal{L}(G)\geq 0, \quad \nabla\tilde{F} = -\frac{\tilde{F}}{\psi}\nabla\psi$$

and by use of (2.13), we have

$$\begin{aligned}
0 \leq & \mathcal{L}(G) = s\psi\mathcal{L}(\tilde{F}) - s(p-1)v\tilde{F}\Delta_\phi\psi - 2s(p-1)v\nabla\tilde{F}\nabla\psi + \psi\tilde{F} \\
= & s\psi\mathcal{L}(\tilde{F}) - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} \\
\leq & s\psi\left(-\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2MK\frac{|\nabla v|^2}{v} + 2p\nabla v\nabla\tilde{F}\right) \\
& -(p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} \\
\leq & -\frac{s\psi}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2s\psi MK\frac{|\nabla v|^2}{v} + 2\frac{p}{(p-1)^{\frac{1}{2}}}M^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi} \\
& -(p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s}.
\end{aligned} \tag{2.14}$$

Applying

$$[(p-1)\Delta_\phi v]^2 = \frac{1}{\alpha^2}\tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2}\tilde{F}\frac{|\nabla v|^2}{v} + \left(\frac{\alpha-1}{\alpha}\right)^2\frac{|\nabla v|^4}{v^2}$$

into (2.14), we obtain

$$\begin{aligned}
0 \leq & -\frac{1}{\tilde{a}s\alpha^2}G^2 - \frac{2(\alpha-1)\psi}{\tilde{a}\alpha^2}G\frac{|\nabla v|^2}{v} - \frac{s\psi^2}{\tilde{a}}\left(\frac{\alpha-1}{\alpha}\right)^2\frac{|\nabla v|^4}{v^2} \\
& + 2s\psi^2 MK\frac{|\nabla v|^2}{v} + 2\frac{p}{(p-1)^{\frac{1}{2}}}M^{\frac{1}{2}}\psi^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi^{\frac{1}{2}}} \\
& -(p-1)v(\Delta_\phi\psi)G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s}.
\end{aligned} \tag{2.15}$$

By virtue of the inequality $-Ax^2+Bx\leq\frac{B^2}{4A}$, we have

$$-\frac{s\psi^2}{\tilde{a}}\left(\frac{\alpha-1}{\alpha}\right)^2\frac{|\nabla v|^4}{v^2} + 2s\psi^2MK\frac{|\nabla v|^2}{v} \leq \frac{\tilde{a}\alpha^2 s\psi^2M^2K^2}{(\alpha-1)^2},$$

$$-\frac{2(\alpha-1)\psi}{\tilde{a}\alpha^2}G\frac{|\nabla v|^2}{v} + 2\frac{p}{(p-1)^{\frac{1}{2}}}M^{\frac{1}{2}}\psi^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi^{\frac{1}{2}}} \leq \frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{|\nabla\psi|^2}{\psi}G.$$

Hence, (2.15) yields

$$\begin{aligned}
0 \leq & -\frac{1}{\tilde{a}s\alpha^2}G^2+\frac{\tilde{a}\alpha^2 s\psi^2M^2K^2}{(\alpha-1)^2}+\frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{|\nabla\psi|^2}{\psi}G\\
& -(p-1)v(L\psi)G+2(p-1)v\frac{|\nabla\psi|^2}{\psi}G+\frac{\psi G}{s}\\
\leq & -\frac{1}{\tilde{a}s\alpha^2}G^2+\left\{\frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{C}{R^2}+(p-1)M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)+\frac{\psi}{s}\right\}G\\
& +\frac{\tilde{a}\alpha^2 s\psi^2M^2K^2}{(\alpha-1)^2}.
\end{aligned} \tag{2.16}$$

Solving the quadratic inequality of G in (2.16) yields

$$\begin{aligned}
G \leq & \frac{\tilde{a}s\alpha^2}{2}\left\{\Big[\frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{C}{R^2}+M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)+\frac{\psi}{s}\Big]\right.\\
& +\left[\Big[\frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{C}{R^2}+M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)+\frac{\psi}{s}\Big]^2\right.\\
& \left.\left.+\frac{4\psi^2M^2K^2}{(\alpha-1)^2}\right]^{\frac{1}{2}}\right\}\\
\leq & \tilde{a}s\alpha^2\left\{\frac{\tilde{a}\alpha^2p^2M}{2(p-1)(\alpha-1)}\frac{C}{R^2}+M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)+\frac{\psi}{s}+\frac{\psi MK}{(\alpha-1)}\right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
G(x,T) \leq & G(x_0,s)\\
\leq & \tilde{a}T\alpha^2\frac{C(m)}{R^2}\left\{\frac{\alpha^2}{(p-1)(\alpha-1)}\tilde{a}p^2M+M\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right\}\\
& +\frac{\alpha^2}{(\alpha-1)}\tilde{a}TMK+\tilde{a}\alpha^2.
\end{aligned} \tag{2.17}$$

For all $x \in B_p(R)$, from (2.17), it holds that

$$\begin{aligned}
F(x,T) \leq & \tilde{a}\alpha^2M\frac{C(m)}{R^2}\left\{\frac{\alpha^2}{\alpha-1}\frac{\tilde{a}p^2}{p-1}+\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right\}\\
& +\frac{\alpha^2}{(\alpha-1)}\tilde{a}MK+\frac{\tilde{a}\alpha^2}{T}.
\end{aligned}$$

Since T is arbitrary, we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. When $p \in (0,1)$ we have $v < 0$ and from (2.3)

$$\begin{aligned}
\mathcal{L}(-\tilde{F}) \leq & \frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2+2(p-1)\mathrm{Ric}_\phi^m(\nabla v,\nabla v)+2p\nabla v\nabla(-\tilde{F})\\
& -(1-\alpha)\left(\frac{v_t}{v}\right)^2\\
\leq & \frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2+2MK\frac{|\nabla v|^2}{-v}+2p\nabla v\nabla(-\tilde{F})\\
& -(1-\alpha)\left(\frac{v_t}{v}\right)^2.
\end{aligned} \tag{2.18}$$

Define $G = t\psi(-\tilde{F})$. Next we will apply maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point (x_0, s), it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla(-\tilde{F}) = -\frac{-\tilde{F}}{\psi}\nabla\psi$$

and by use of (2.18), we have

$$\begin{aligned}
0 \leq & \mathcal{L}(G) = s\psi\mathcal{L}(-\tilde{F}) - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} \\
\leq & s\psi\left(\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2MK\frac{|\nabla v|^2}{-v} + 2p\nabla v\nabla(-\tilde{F})\right) \\
& - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} - (1-\alpha)s\psi\left(\frac{v_t}{v}\right)^2 \\
\leq & \frac{s\psi}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2s\varphi MK\frac{|\nabla v|^2}{-v} + 2\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}G\frac{|\nabla v|}{(-v)^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi} \\
& - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} - (1-\alpha)s\psi\left(\frac{v_t}{v}\right)^2.
\end{aligned} \tag{2.19}$$

Applying

$$[(p-1)\Delta_\phi v]^2 = \frac{1}{\alpha^2}\tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2}\tilde{F}\frac{|\nabla v|^2}{v} + \left(\frac{\alpha-1}{\alpha}\right)^2\frac{|\nabla v|^4}{v^2},$$

$$\left(\frac{v_t}{v}\right)^2 = \frac{1}{\alpha^2}\left(-\tilde{F} + \frac{|\nabla v|^2}{v}\right)^2 = \frac{1}{\alpha^2}(-\tilde{F})^2 + \frac{2}{\alpha^2}(-\tilde{F})\frac{|\nabla v|^2}{v} + \frac{1}{\alpha^2}\frac{|\nabla v|^4}{v^2}$$

into (2.19), we obtain

$$\begin{aligned}
0 \leq & \frac{1}{\tilde{a}s\alpha^2}\Big\{[1-\tilde{a}(1-\alpha)]G^2 - 2(1-\tilde{a})(1-\alpha)s\psi G\frac{|\nabla v|^2}{-v} \\
& + s^2\psi^2(1-\alpha)(1-\alpha-\tilde{a})\frac{|\nabla v|^4}{v^2}\Big\} + 2s\psi^2 MK\frac{|\nabla v|^2}{-v} \\
& + 2\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}\psi^{\frac{1}{2}}G\frac{|\nabla v|}{(-v)^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi^{\frac{1}{2}}} - (p-1)v(\Delta_\phi\psi)G \\
& + 2(p-1)v\frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s}.
\end{aligned} \tag{2.20}$$

Next we take the similar method as in Theorem 4.1 of [20]. Since $p \in (1-\frac{2}{m}, 1)$, we have $\tilde{a} < 0$. Thus, we have for any positive constants $\varepsilon_1, \varepsilon_2$,

$$2s\psi^2 MK\frac{|\nabla v|^2}{-v} \leq -\varepsilon_1\frac{s^2\psi^2}{\tilde{a}s\alpha^2}(1-\alpha)(1-\alpha-\tilde{a})\frac{|\nabla v|^4}{v^2} - \frac{1}{\varepsilon_1}\frac{\tilde{a}s\alpha^2(p-1)^2\psi^2M^2K^2}{(1-\alpha)(1-\alpha-\tilde{a})},$$

$$\begin{aligned}
2\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}\psi^{\frac{1}{2}}G\frac{|\nabla v|}{(-v)^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi^{\frac{1}{2}}} \leq & -\varepsilon_2\frac{2}{\tilde{a}s\alpha^2}(1-\tilde{a})(1-\alpha)s\psi G\frac{|\nabla v|^2}{-v} \\
& - \frac{\tilde{a}\alpha^2p^2M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)}\frac{|\nabla\psi|^2}{\psi}G.
\end{aligned}$$

Hence, we get from (2.20) that

$$
\begin{aligned}
0 \leq & -\frac{1}{\tilde{a}s\alpha^2}\Big\{-[1-\tilde{a}(1-\alpha)]G^2 + 2(1+\varepsilon_2)(1-\tilde{a})(1-\alpha)s\psi G\frac{|\nabla v|^2}{-v} \\
& -(1-\varepsilon_1)s^2\psi^2(1-\alpha)(1-\alpha-\tilde{a})\frac{|\nabla v|^4}{v^2}\Big\} - \frac{1}{\varepsilon_1}\frac{as\alpha^2\psi^2M^2K^2}{(1-\alpha)(1-\alpha-\tilde{a})} \\
& -\frac{\tilde{a}\alpha^2p^2M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)}\frac{|\nabla\psi|^2}{\psi}G - (p-1)v(\Delta_\phi\psi)G \\
& + 2(p-1)v\frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s} \\
\leq & \frac{1}{\tilde{a}s\alpha^2}\Big\{[1-\tilde{a}(1-\alpha)] - \frac{(1+\varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1-\varepsilon_1)(1-\alpha-\tilde{a})}\Big\}G^2 - \frac{1}{\varepsilon_1}\frac{\tilde{a}s\alpha^2\psi^2M^2K^2}{(1-\alpha)(1-\alpha-\tilde{a})} \\
& -\frac{\tilde{a}\alpha^2p^2M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)}\frac{|\nabla\psi|^2}{\psi}G - (p-1)v(\Delta_\phi\psi)G \\
& + 2(p-1)v\frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s}.
\end{aligned} \tag{2.21}
$$

Taking $\varepsilon_1, \varepsilon_2$ such that

$$
[1-\tilde{a}(1-\alpha)] - \frac{(1+\varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1-\varepsilon_1)(1-\alpha-\tilde{a})} := A(\varepsilon_1,\varepsilon_2) > 0, \tag{2.22}
$$

then (2.21) yields

$$
\begin{aligned}
0 \leq & -\frac{1}{(-\tilde{a})s\alpha^2}A(\varepsilon_1,\varepsilon_2)G^2 + \Big\{\frac{(-\tilde{a})\alpha^2p^2M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)}\frac{C}{R^2} \\
& + M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right) + \frac{\psi}{s}\Big\}G + \frac{(-\tilde{a})s\alpha^2\psi^2M^2K^2}{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})}.
\end{aligned} \tag{2.23}
$$

Solving the quadratic inequality of G in (2.23) yields

$$
\begin{aligned}
G \leq & \frac{(-\tilde{a})s\alpha^2}{A(\varepsilon_1,\varepsilon_2)}\left\{\frac{(-\tilde{a})\alpha^2p^2M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)}\frac{C}{R^2} + M\frac{C(m)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right) + \frac{\psi}{s}\right. \\
& \left. + \frac{\psi MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})}}\sqrt{A(\varepsilon_1,\varepsilon_2)}\right\}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
G(x,T) \leq & G(x_0,s) \\
\leq & \frac{(-\tilde{a})T\alpha^2M}{A(\varepsilon_1,\varepsilon_2)}\frac{C(m)}{R^2}\left\{\frac{(-\tilde{a})\alpha^2p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + \left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right\} \\
& + \frac{(-\tilde{a})T\alpha^2MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})A(\varepsilon_1,\varepsilon_2)}} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1,\varepsilon_2)}.
\end{aligned} \tag{2.24}
$$

and for $x \in B_p(R)$,

$$
\begin{aligned}
-F(x,t) \leq & \frac{(-\tilde{a})\alpha^2M}{A(\varepsilon_1,\varepsilon_2)}\frac{C(m)}{R^2}\left\{\frac{(-\tilde{a})\alpha^2p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + \left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right\} \\
& + \frac{(-\tilde{a})\alpha^2MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})A(\varepsilon_1,\varepsilon_2)}} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1,\varepsilon_2)t}.
\end{aligned}
$$

This completes the proof of Theorem 1.2.

3 Proofs of Theorem 1.3-1.7

Under the assumption that $\mathrm{Ric}_\phi^m \geq -K$ and $p > 1$, (2.2) shows that

$$\begin{aligned}\mathcal{L}(F) \leq & -\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2(p-1)K|\nabla v|^2 + 2p\nabla v\nabla F \\ & + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha'\frac{v_t}{v} - \varphi' \\ \leq & -\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2MK\frac{|\nabla v|^2}{v} + 2p\nabla v\nabla F \\ & + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha'\frac{v_t}{v} - \varphi'.\end{aligned} \tag{3.1}$$

Following the methods in [13], we can prove that Theorem 1.3, 1.5, 1.6, 1.7 hold respectively.

Next we are in a position to prove Theorem 1.4. Define $\overline{F} = \frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}$, where $0 < \alpha < 1$ is a constant. Then (2.3) shows that

$$\begin{aligned}\mathcal{L}(-\overline{F}) \leq & \frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 + 2MK\frac{|\nabla v|^2}{-v} + 2p\nabla v\nabla(-\overline{F}) - (1-\alpha)\left(\frac{v_t}{v}\right)^2 \\ = & \frac{1}{\tilde{a}\alpha^2}\Big(-\overline{F} - (1-\alpha)\frac{|\nabla v|^2}{-v}\Big)^2 + 2MK\frac{|\nabla v|^2}{-v} + 2p\nabla v\nabla(-\overline{F}) \\ & - \frac{1-\alpha}{\alpha^2}\Big(-\overline{F} - \frac{|\nabla v|^2}{-v}\Big)^2.\end{aligned} \tag{3.2}$$

Let $G = t\psi(-\overline{F})$. We apply maximum principle to G on $B_p(2R) \times [0, T]$ and assume that G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ with $G(x_0, s) > 0$ (otherwise the proof is trivial). Then at the point (x_0, s), it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla(-\overline{F}) = -\frac{-\overline{F}}{\psi}\nabla\psi$$

and by use of (3.2), we have

$$\begin{aligned}0 \leq & \mathcal{L}(G) = s\psi\mathcal{L}(-\overline{F}) - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s} \\ \leq & \frac{s\psi}{\tilde{a}\alpha^2}\Big(-\overline{F} - (1-\alpha)\frac{|\nabla v|^2}{-v}\Big)^2 + 2s\varphi MK\frac{|\nabla v|^2}{-v} + 2\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}G\frac{|\nabla v|}{(-v)^{\frac{1}{2}}}\frac{|\nabla\psi|}{\psi} \\ & - \frac{1-\alpha}{\alpha^2}s\psi\Big(-\overline{F} - \frac{|\nabla v|^2}{-v}\Big)^2 - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s}.\end{aligned}$$

Let $\frac{|\nabla v|^2}{-v} = \mu(-\overline{F})$ at the point (x_0, s). Then we have $\mu \geq 0$ and

$$\begin{aligned}0 \leq & \frac{1}{\tilde{a}\alpha^2 s\psi}[1 - (1-\alpha)\mu]^2G^2 + 2\mu MKG + \frac{2\mu^{\frac{1}{2}}}{s^{\frac{1}{2}}\psi^{\frac{1}{2}}}\frac{p}{(1-p)^{\frac{1}{2}}}M^{\frac{1}{2}}G^{\frac{3}{2}}\frac{|\nabla\psi|}{\psi} \\ & - \frac{1-\alpha}{\alpha^2}\frac{1}{s\psi}(1-\mu)^2G^2 - (p-1)v\frac{\Delta_\phi\psi}{\psi}G + 2(p-1)v\frac{|\nabla\psi|^2}{\psi^2}G + \frac{G}{s}.\end{aligned} \tag{3.3}$$

Multiplying the both sides of (3.3) by $\frac{s\psi}{G}$ yields

$$\begin{aligned}0 \leq & \frac{1}{\tilde{a}\alpha^2}[1-(1-\alpha)\mu]^2 G + 2\mu MKs\psi + 2\mu^{\frac{1}{2}} s^{\frac{1}{2}} \frac{p}{(1-p)^{\frac{1}{2}}} M^{\frac{1}{2}} \frac{|\nabla\psi|}{\psi^{\frac{1}{2}}} G^{\frac{1}{2}} \\ & - \frac{1-\alpha}{\alpha^2}(1-\mu)^2 G - (p-1)sv\Delta_\phi\psi + 2(p-1)sv\frac{|\nabla\psi|^2}{\psi} + \psi \\ = & -\tilde{A}G + 2\tilde{B}G^{\frac{1}{2}} + \tilde{C},\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\tilde{A} &= \frac{1}{-\tilde{a}\alpha^2}[1-(1-\alpha)\mu]^2 + \frac{1-\alpha}{\alpha^2}(1-\mu)^2, \\ \tilde{B} &= \mu^{\frac{1}{2}} s^{\frac{1}{2}} \frac{p}{(1-p)^{\frac{1}{2}}} M^{\frac{1}{2}} \frac{|\nabla\psi|}{\psi^{\frac{1}{2}}}, \\ \tilde{C} &= 2\mu MKs\psi + (1-p)s(-v)\Big(-\Delta_\phi\psi + 2\frac{|\nabla\psi|^2}{\psi}\Big) + \psi.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\frac{1}{\tilde{A}} = & \frac{(-\tilde{a})\alpha^2}{[1-(1-\alpha)\mu]^2 + (-\tilde{a})(1-\alpha)(1-\mu)^2} \\ = & \frac{(-\tilde{a})\alpha^2}{1+(-\tilde{a})(1-\alpha) - 2(1-\alpha)(1-\tilde{a})\mu + (1-\alpha)(1-\alpha-\tilde{a})\mu^2} \\ \leq & 1-\alpha-\tilde{a},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{2\mu}{\tilde{A}} = & \frac{2(-\tilde{a})\alpha^2\mu}{1+(-\tilde{a})(1-\alpha) - 2(1-\alpha)(1-\tilde{a})\mu + (1-\alpha)(1-\alpha-\tilde{a})\mu^2} \\ \leq & \frac{(-\tilde{a})\alpha^2}{\sqrt{[1+(-\tilde{a})(1-\alpha)](1-\alpha)(1-\alpha-\tilde{a})} - (1-\alpha)(1-\tilde{a})} \\ = & \sqrt{[\frac{1}{1-\alpha} + (-\tilde{a})](1-\alpha-\tilde{a})} + (1-\tilde{a}) \\ \leq & \frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}),\end{aligned} \tag{3.6}$$

where the last inequality used $\sqrt{xy} \leq \frac{1}{2}(x+y)$ and there exists a constant $C(\tilde{a},\alpha)$ such that $\frac{\mu^{\frac{1}{2}}}{\tilde{A}} \leq C(\tilde{a},\alpha)$. From the inequality $\tilde{A}x^2 - 2\tilde{B}x \leq \tilde{C}$, we have $x \leq \frac{2\tilde{B}}{\tilde{A}} + \sqrt{\frac{\tilde{C}}{\tilde{A}}}$. Applying this inequality into (3.4) by letting $x = G^{\frac{1}{2}}$ gives

$$\begin{aligned}G^{\frac{1}{2}} \leq & C(\tilde{a},\alpha) s^{\frac{1}{2}} \frac{p}{(1-p)^{\frac{1}{2}}} M^{\frac{1}{2}} \frac{C}{R} + \Big[\Big(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a})\Big) MKs + 1-\alpha-\tilde{a} \\ & + (1-p)(1-\alpha-\tilde{a})Ms\frac{C(m)}{R^2}\Big(1+\sqrt{K}R\coth(\sqrt{K}R)\Big)\Big]^{\frac{1}{2}}\end{aligned} \tag{3.7}$$

Hence, for $x \in B_p(R)$, we have

$$\begin{aligned}-\frac{|\nabla v|^2}{v} + \alpha\frac{v_t}{v} \leq & \Bigg\{ C(\tilde{a},\alpha)\frac{p}{(1-p)^{\frac{1}{2}}} M^{\frac{1}{2}} \frac{C}{R} + \Big[\Big(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a})\Big) MK + \frac{1-\alpha-\tilde{a}}{t} \\ & + (1-p)(1-\alpha-\tilde{a})M\frac{C(m)}{R^2}\Big(1+\sqrt{K}R\coth(\sqrt{K}R)\Big)\Big]^{\frac{1}{2}} \Bigg\}^2.\end{aligned} \tag{3.8}$$

We complete the proof of Theorem 1.4.

4 Proofs of Theorem 1.8 and 1.9

Lemma 4.1. *If M^n is a compact Riemannian manifold and u is a positive solution to* (1.8) *with $p \neq 0$, then*

$$\frac{d}{dt}\int_{M^n} uv\,d\mu = (p-1)\int_{M^n}(\Delta_\phi v)uv\,d\mu = -p\int_{M^n}|\nabla v|^2 u\,d\mu. \tag{4.1}$$

Proof. From (2.1), we have $(uv)_t = vu_t + uv_t = v\Delta_\phi(u^p) + (p-1)uv\Delta_\phi v + u|\nabla v|^2$. It follows from $\nabla(u^p) = u\nabla v$ that

$$\int_{M^n}[v\Delta_\phi(u^p) + u|\nabla v|^2]\,d\mu = \int_{M^n}[-\nabla v\nabla(u^p) + u|\nabla v|^2]\,d\mu = 0.$$

Hence

$$\begin{aligned}\frac{d}{dt}\int_{M^n} uv\,d\mu &= \int_{M^n}(uv)_t\,d\mu\\ &= \int_{M^n}[v\Delta_\phi(u^p) + (p-1)uv\Delta_\phi v + u|\nabla v|^2]\,d\mu\\ &=(p-1)\int_{M^n}(\Delta_\phi v)uv\,d\mu\\ &=p\int_{M^n}(\Delta_\phi v)u^p\,d\mu\\ &=-p\int_{M^n}\nabla v\nabla(u^p)\,d\mu\\ &=-p\int_{M^n}|\nabla v|^2 u\,d\mu.\end{aligned}$$

We complete the proof of Lemma 4.1. □

Lemma 4.2. *If M^n is a compact Riemannian manifold and u is a positive solution to* (1.8) *with $p \neq 0$, then*

$$\frac{d}{dt}\int_{M^n}(\Delta_\phi v)uv\,d\mu = 2\int_{M^n}\Big[(p-1)(\Delta_\phi v)^2 + |\nabla^2 v|^2 + \mathrm{Ric}_\phi(\nabla v, \nabla v)\Big]uv\,d\mu. \tag{4.2}$$

Proof. Noticing

$$\frac{d}{dt}\int_{M^n}(\Delta_\phi v)uv\,d\mu = \int_{M^n}[(\Delta_\phi v)_t uv + (\Delta_\phi v)(uv)_t]\,d\mu. \tag{4.3}$$

A direct calculation gives

$$\begin{aligned}(\Delta_\phi v)_t =&\Delta_\phi[(p-1)v\Delta_\phi v+|\nabla v|^2]\\ =&(p-1)[(\Delta_\phi v)^2+2\nabla v\nabla\Delta_\phi v+v\Delta_\phi^2 v]+\Delta_\phi|\nabla v|^2\\ =&(p-1)(\Delta_\phi v)^2+2p\nabla v\nabla\Delta_\phi v+(p-1)v\Delta_\phi^2 v+2[|\nabla^2 v|^2+\mathrm{Ric}_\phi(\nabla v,\nabla v)].\end{aligned}$$

We derive from $(p-1)\nabla(uv^2)=(2p-1)uv\nabla v$ that

$$\begin{aligned}&\int_{M^n}[2p\nabla v\nabla\Delta_\phi v+(p-1)v\Delta_\phi^2 v]uv\,d\mu\\ &=\int_{M^n}2p\nabla v\nabla(\Delta_\phi v)uv\,d\mu-\int_{M^n}(p-1)\nabla(uv^2)\nabla\Delta_\phi v\,d\mu\\ &=\int_{M^n}\nabla v\nabla(\Delta_\phi v)uv\,d\mu.\end{aligned}$$

Hence,

$$\int_{M^n}(\Delta_\phi v)_t uv\,d\mu=\int_{M^n}\Big\{(p-1)(\Delta_\phi v)^2+\nabla v\nabla\Delta_\phi v+2[|\nabla^2 v|^2+\mathrm{Ric}_\phi(\nabla v,\nabla v)]\Big\}uv\,d\mu. \tag{4.4}$$

On the other hand,

$$\begin{aligned}\int_{M^n}\Delta_\phi v(uv)_t\,d\mu=&\int_{M^n}\Delta_\phi v[v\Delta_\phi(u^p)+(p-1)uv\Delta_\phi v+u|\nabla v|^2]\,d\mu\\ =&\int_{M^n}[-\nabla(v\Delta_\phi v)\nabla(u^p)+(p-1)uv(\Delta_\phi v)^2+u|\nabla v|^2\Delta_\phi v]\,d\mu\\ =&\int_{M^n}[-\nabla(v\Delta_\phi v)u\nabla v+(p-1)uv(\Delta_\phi v)^2+u|\nabla v|^2\Delta_\phi v]\,d\mu\\ =&\int_{M^n}[-\nabla v\nabla\Delta_\phi v+(p-1)(\Delta_\phi v)^2]uv\,d\mu.\end{aligned} \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3) concludes the proof of Lemma 4.2. □

Proof of Theorem 1.8 and 1.9. By Lemma 4.1, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)=&-\tilde a t^{\tilde a-1}\int_{M^n}uv\,d\mu-(p-1)t^{\tilde a}\int_{M^n}(\Delta_\phi v)uv\,d\mu\\ =&-t^{\tilde a}\int_{M^n}\left((p-1)\Delta_\phi v+\frac{\tilde a}{t}\right)uv\,d\mu.\end{aligned}$$

We obtain (1.40) and (1.43). On the other hand, from the definition of $\mathcal{W}_{p,m}(g,u,t)$ in (1.39), we have

$$\begin{aligned}\mathcal{W}_{p,m}(g,u,t)=&\frac{d}{dt}[t\mathcal{N}_{p,m}(g,u,t)]\\ =&\mathcal{N}_{p,m}(g,u,t)+t\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)\\ =&t^{\tilde a+1}\int_{M^n}\Big(p\frac{|\nabla v|^2}{v}-\frac{\tilde a+1}{t}\Big)uv\,d\mu,\end{aligned}$$

where the Lemma 4.1 was used in the last equality. Hence, we derive (1.41) and (1.44).

Noticing that the estimate (1.10) also holds for compact Riemannian manifolds. Taking $K = 0$ and then letting $\alpha \to 1$ in (1.10) yields

$$(p-1)\Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \geq 0,$$

which concludes that if $\mathrm{Ric}_\phi^m \geq 0$, then $\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t) \leq 0$ and $\mathcal{N}_{p,m}(g,u,t)$ is a monotone non-increasing in t. When $p \in (1-\frac{2}{m}, 1)$ and $\mathrm{Ric}_\phi^m \geq 0$, we also get from (1.12) that

$$(p-1)\Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \leq 0,$$

which shows that $\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t) \leq 0$ and $\mathcal{N}_{p,m}(g,u,t)$ is also a monotone non-increasing in t.

Next we are in a position to prove (1.42). From (1.40), we have

$$\begin{aligned}
&\frac{d}{dt}[t\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t)]\\
&\quad=\frac{d}{dt}\Big[-t^{\tilde{a}+1}\int_{M^n}(p-1)(\Delta_\phi v)uv\,d\mu - \tilde{a}t^{\tilde{a}}\int_{M^n}uv\,d\mu\Big]\\
&\quad=\frac{d}{dt}\Big[-t^{\tilde{a}+1}\int_{M^n}(p-1)(\Delta_\phi v)uv\,d\mu + \tilde{a}\mathcal{N}_{p,m}(g,u,t)\Big]\\
&\quad=-2t^{\tilde{a}+1}\int_{M^n}\Big[(p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\mathrm{Ric}_\phi(\nabla v,\nabla v)\Big]uv\,d\mu\\
&\qquad-(\tilde{a}+1)t^{\tilde{a}}\int_{M^n}(p-1)(\Delta_\phi v)uv\,d\mu - \tilde{a}t^{\tilde{a}}\int_{M^n}\Big((p-1)\Delta_\phi v + \frac{\tilde{a}}{t}\Big)uv\,d\mu,
\end{aligned}$$

where the last equality used the Lemma 4.2. Hence,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t)\\
&=\frac{d}{dt}[t\frac{d}{dt}\mathcal{N}_{p,m}(g,u,t) + \mathcal{N}_{p,m}(g,u,t)]\\
&=-2t^{\tilde{a}+1}\int_{M^n}\Big[(p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\mathrm{Ric}_\phi(\nabla v,\nabla v)\Big]uv\,d\mu\\
&\quad-(\tilde{a}+1)t^{\tilde{a}}\int_{M^n}(p-1)(\Delta_\phi v)uv\,d\mu - (\tilde{a}+1)t^{\tilde{a}}\int_{M^n}\Big((p-1)\Delta_\phi v + \frac{\tilde{a}}{t}\Big)uv\,d\mu\\
&=-2t^{\tilde{a}+1}\int_{M^n}\Big[(p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\mathrm{Ric}_\phi(\nabla v,\nabla v)\\
&\quad+(p-1)\frac{\tilde{a}+1}{t}\Delta_\phi v + \frac{\tilde{a}^2+\tilde{a}}{2t^2}\Big]uv\,d\mu.
\end{aligned} \tag{4.6}$$

Noticing

$$\begin{aligned}
&(p-1)^2(\Delta_\phi v)^2 + (p-1)\frac{\tilde{a}+1}{t}\Delta_\phi v + \frac{\tilde{a}^2+\tilde{a}}{2t^2}\\
=&\Big|(p-1)\Delta_\phi v + \frac{m(p-1)}{[m(p-1)+2]t}\Big|^2 + \frac{2(p-1)}{[m(p-1)+2]t}\Delta_\phi v + \frac{(p-1)m}{[m(p-1)+2]^2t^2},
\end{aligned}$$

and hence

$$
\begin{aligned}
&(p-1)^2(\Delta_\phi v)^2+(p-1)\frac{\tilde a+1}{t}\Delta_\phi v+\frac{\tilde a^2+\tilde a}{2t^2}+(p-1)|\nabla^2 v|^2+\frac{p-1}{m-n}(\nabla\phi\nabla v)^2\\
=&\Big|(p-1)\Delta_\phi v+\frac{m(p-1)}{[m(p-1)+2]t}\Big|^2\\
&+(p-1)\Big|\nabla^2 v+\frac{g}{[m(p-1)+2]t}\Big|^2+\frac{p-1}{m-n}\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2.
\end{aligned}\tag{4.7}
$$

We complete the proof of (1.42) by putting (4.7) into (4.6).

When $p\in(0,1)$, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
-(p-1)&\Big|\nabla^2 v+\frac{g}{[m(p-1)+2]t}\Big|^2\\
&\geq-\frac{p-1}{n}\Big|\Delta v+\frac{n}{[m(p-1)+2]t}\Big|^2\\
&=-\frac{1}{n(p-1)}\Big|(p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big|^2-\frac{p-1}{n}\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2\\
&\quad-\frac{2}{n}\Big((p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big)\Big(\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
-(p-1)&\Big|\nabla^2 v+\frac{g}{[m(p-1)+2]t}\Big|^2-\frac{p-1}{m-n}\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2\\
&-\Big|(p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big|^2\\
\geq&\frac{1-n(1-p)}{n(1-p)}\Big|(p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big|^2+\frac{m(1-p)}{n(m-n)}\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2\\
&-\frac{2}{n}\Big((p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big)\Big(\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big)\\
\geq&\Big(\frac{1-n(1-p)}{n(1-p)}-\frac{\varepsilon}{n}\Big)\Big|(p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big|^2\\
&+\Big(\frac{m(1-p)}{n(m-n)}-\frac{1}{n\varepsilon}\Big)\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2,
\end{aligned}\tag{4.8}
$$

where $\varepsilon\geq m-n$ is a positive constant and satisfies $1-\frac{1}{n+\varepsilon}\leq p\leq 1-\frac{m-n}{m\varepsilon}$. Inserting (4.8) into (1.42) gives

$$
\begin{aligned}
\frac{d}{dt}\mathcal{W}_{p,m}(g,u,t)\leq&2t^{a+1}\int\limits_{M^n}\Bigg\{(1-p)\mathrm{Ric}_\phi^m(\nabla v,\nabla v)\\
&+\Big(\frac{1-n(1-p)}{n(1-p)}-\frac{\varepsilon}{n}\Big)\Big|(p-1)\Delta_\phi v+\frac{\tilde a}{t}\Big|^2\\
&+\Big(\frac{m(1-p)}{n(m-n)}-\frac{1}{n\varepsilon}\Big)\Big|\nabla\phi\nabla v-\frac{m-n}{[m(p-1)+2]t}\Big|^2\Bigg\}uv\,d\mu.
\end{aligned}\tag{4.9}
$$

Therefore, we complete the proof of (1.45).

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